Неодређени интеграл

$f'(x) = 4x^3 - 2\sin 2x$

$\underline{f(x) = ?}$

$x^4 + \cos 2x + C$

$$\int (4x^3 - 2\sin 2x)dx = x^4 + \cos 2x + C$$

$$\int \underbrace{f(x)}_{\substack{\text{подин-}\\\text{тегрална}\\\text{ф-ја}}} dx = \underbrace{F(x)}_{\substack{\text{прими-}\\\text{тивна}\\\text{ф-ја}}} + C \quad \text{акко} \quad F'(x) = f(x)$$

$$\boxed{\int e^{-x^2} dx, \qquad \int \frac{\sin x}{x} dx} \quad \text{не могу да се реше}$$

Таблице

(1) $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \qquad \alpha \neq -1$

(2) $\int \frac{1}{x} dx = \ln|x| + C$

(3) $\int a^x dx = \frac{a^x}{\ln a} + C, \qquad a > 0$

(3′) $\int e^x dx = e^x + C, \qquad a > 0$

(4) $\int \sin x \, dx = -\cos x + C$

(5) $\int \cos x \, dx = \sin x + C$

(6) $\int \frac{1}{\cos^2 x} dx = tg\, x + C$

(7) $\int \frac{1}{\sin^2 x} dx = -ctg\, x + C$

(8) $\int \frac{1}{1+x^2} dx = arctg\, x + C$

(8′) $\int \frac{1}{a^2+x} dx = \frac{1}{a} arctg \frac{x}{a} + C$

(9) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

(9′) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$

(10) $\int \frac{1}{\sqrt{2}} dx = 2\sqrt{2} + C$

(11) $\int \frac{1}{x^\alpha} dx = -\frac{1}{(\alpha-1)x^{\alpha-1}} + C$

(12) $\int \frac{1}{\sqrt{x^2+1}} dx = \ln\left(x + \sqrt{x^2+1}\right) + C$

Основне особине неодређеног интеграла

(I) $\int [u(x) \pm v(x)]dx = \int u(x)dx \pm \int v(x)dx$

$(II)\ \int a\cdot u(x)dx = a\int u(x)dx$

Непосредна интеграција

1) $\int \frac{x^2-3x+4}{\sqrt[3]{x}}dx = \int\left(x^{\frac{5}{3}}-3x^{\frac{2}{3}}+4x^{-\frac{1}{3}}\right)dx = \int x^{\frac{5}{3}}dx - 3\int x^{\frac{2}{3}}dx + 4\int x^{-\frac{1}{3}}dx =$

$= \frac{x^{\frac{8}{3}}}{\frac{8}{3}} - 3\cdot\frac{x^{\frac{5}{3}}}{\frac{5}{3}} + 4\cdot\frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{3}{8}x^2\sqrt[3]{x^2} - \frac{9}{5}x\sqrt[3]{x^2} + 6\sqrt[3]{x^2} + C$

2) $\int \frac{x^2}{x^2+1}dx = \int\frac{x^2+1-1}{x^2+1}dx = \int\left(1-\frac{1}{x^2+1}\right)dx = \int 1dx - \int\frac{1}{x^2+1}dx =$
$= x - arctg\ x + C$

3) $\int tg^2x dx = \int\frac{\sin^2 x}{\cos^2 x}dx = \int\frac{1-\cos^2 x}{\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x}-1\right)dx = \int\frac{1}{\cos^2 x}dx - \int 1dx$
$= tg\ x - x + C$

4) $\int\frac{1}{\sin^2 x\cdot\cos^2 x}dx = \int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cdot\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right)dx = tg\ x - ctg\ x + C$

5) $\int(x-2)^3dx = \int(x^3-6x^2+12x-8)dx = \int x^3dx - 6\int x^2dx + 12\int x\,dx - 8\int dx$

$= \frac{1}{4}x^4 - 6\cdot\frac{x^3}{3} + 12\cdot\frac{x^2}{2} - 8x + C = \frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x + C$

6) $\int(x-2)^3dx = \frac{1}{4}(x-2)^4 + C$

7) $[\text{колок. }2]\int(3x^2-e^{-x})dx = x^3 + e^{-x} + C$

8) $[\text{колок. }2]\int(2x^4-\sin(\pi x))dx = 2\cdot\frac{1}{5}x^5 - \frac{1}{\pi}\cdot\cos(\pi x) + C$

9) $\int e^{\frac{x}{4}-1}dx = 4\cdot e^{\frac{x}{4}-1} + C$

10) $\int\frac{1}{1-2x}dx = -\frac{1}{2}\ln|1-2x| + C$

Метод замене промењиве

$\int f(x)dx \qquad\qquad I = \int f(g(t))\cdot g'(t)\cdot dt$

$x = g(t) \qquad\qquad I = F\big(g(t)\big) + C$

$t = g'(x)dx = g'(t)dt \qquad I = F\big(g'(x)\big) + C$

11) $\int e^{\frac{x}{4}-1}dx = \int e^t\cdot 4dt = 4\int e^t dt = 4\cdot e^t + C = 4\cdot e^{\frac{x}{4}-1} + C$

$\frac{x}{4}-1 = t \quad \frac{1}{4}dx = dt \quad dx = 4dt$

12) $\int\frac{1}{x^2+a^2}dx \qquad \frac{x}{a} = t \qquad x = at \qquad dx = a\cdot dt$

$I = \int\frac{a\cdot dt}{a^2t^2+a^2} = \frac{1}{a}\int\frac{1}{t^2+1}dx = \frac{1}{a}arctg\ t + C = \frac{1}{a}arctg\frac{x}{a} + C$

13) $[\text{колок. }2]\ I = \int(8x-16)\sin(2x^2-8x+5)\,dx =$

$2x^2-8x+5 = t \qquad (4x-8)dx = dt\ /\cdot 2 \qquad (8x-16)dx = 2dt$

$$I=\int \sin t\cdot 2dt = 2\cdot\cos t + C = -2\cos(2x^2-8x+5)+C$$

14) $$\int \frac{\sin 3x}{\sqrt{2+\cos 3x}}dx = \int -\frac{2}{3}dt = -\frac{2}{3}t + C = -\frac{2}{3}\sqrt{2+\cos 3x}+C$$

$$\sqrt{2+\cos 3x}=t \qquad 2+\cos 3x = t^2 \qquad \frac{1}{2\sqrt{2+\cos 3x}} = -\frac{2}{3}dt$$

15) [испит] $$I=\int \frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}dx = \int \sqrt[3]{\frac{1}{t}}\cdot\frac{1}{2}dt = \frac{1}{2}\cdot\int t^{-\frac{1}{3}}dt = \frac{1}{2}\cdot\frac{t^{\frac{2}{3}}}{\frac{2}{3}}+C=$$

$$\frac{2+x}{2-x}=t \quad \frac{2-x}{2+x}=\frac{1}{t} \quad \frac{2-x-(-1)(2+x)}{(2-x)^2}dx = dt \quad \frac{4dx}{(2-x)^2}=dt \quad \frac{2dx}{(2-x)^2}\frac{1}{2}dt$$

$$=\frac{3}{4}\sqrt[3]{t^2}+C = \frac{3}{4}\cdot\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2}+C$$

16) $$\int \frac{5x+4}{2x^2-3x+8}dx=$$

1! $D=9-64<0$

(1) $$I_1=\int\frac{4x-3}{2x^2-3x+8}dx$$

$$2x^2-3x+8=t \qquad (4x-3)dx=dt$$

$$I_1=\int\frac{dt}{t}=\ln|t|+C=\ln(2x^2-3x+8)+C$$

(2) $$I_2=\int\frac{1}{2x^2-3x+8}dx=\frac{1}{2}\int\frac{1}{x^2-\frac{3}{2}x+4}dx$$

$$f(x)=x^2+6x+7=x^2+6x+\boxed{9}-\boxed{9}+7=\underbrace{(x+3)^2-2}_{\text{канонски облик квадр.ф−је}}$$

$$x^2-\frac{3}{2}x+4=x^2-\frac{3}{2}x+\boxed{\left(\frac{3}{4}\right)^2}-\boxed{\left(\frac{3}{4}\right)^2}+4=\left(x-\frac{3}{4}\right)^2+\left(\frac{\sqrt{55}}{4}\right)^2$$

$$4-\frac{9}{16}=\frac{64-9}{16}=\frac{55}{16}=\left(\frac{\sqrt{55}}{4}\right)^2$$

$$I_2=\int\frac{1}{\left(x-\frac{3}{4}\right)^2+\left(\frac{\sqrt{55}}{4}\right)^2}dx=\frac{1}{2}\cdot\frac{1}{\frac{\sqrt{55}}{4}}arctg\frac{x-\frac{3}{4}}{\frac{\sqrt{55}}{4}}+C_2=$$

$$=\frac{2}{\sqrt{55}}arctg\frac{4x-3}{\sqrt{55}}+C_2$$

$$I=\int\frac{5x+4}{2x^2-3x+8}dx=5\cdot\int\frac{x+\frac{4}{5}}{2x^2-3x+8}dx=\frac{5}{4}\int\frac{4x+\frac{16}{5}}{2x^2-3x+8}dx=$$

$$=\frac{5}{4}\int\frac{4x-3+3+\frac{16}{5}}{2x^2-3x+8}dx=\frac{5}{4}\int\frac{4x-3}{2x^2-3x+8}dx+\frac{5}{4}\int\frac{\frac{31}{5}}{2x^2-3x+8}dx=$$

$$=\frac{5}{4}\int\frac{4x-3}{2x^2-3x+8}dx+\frac{31}{4}\int\frac{1}{2x^2-3x+8}dx=\frac{5}{4}\cdot I_1+\frac{31}{4}\cdot I_2=$$

$$=\frac{5}{4}\ln(2x^2-3x+8)+\frac{31}{2\sqrt{55}}arctg\frac{4x-3}{\sqrt{55}}+C$$

Парцијална интеграција

$$\int udv=uv-\int vdu\,;\quad \begin{matrix}u=u(x)\\ v=v(x)\end{matrix}\ \in D(A)$$

17) [колок. 2] $\int x^2\ln x\,dx=\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2dx=\frac{1}{3}x^3\ln x-\frac{1}{3}\cdot\frac{1}{3}x^3+C=$

$u=\ln x\,, du=\frac{1}{x}dx;\quad x^2dx=dv, \frac{1}{3}x^3=v \qquad\qquad =\frac{1}{3}x^3(3\ln x-1)+C$

18) $\int(x^2-2x)\sin\frac{x}{2}dx=$

$u=x^2-2x,\qquad \sin\frac{x}{2}dx=dv;\quad du=(2x-2)dx,\quad -2\cos\frac{x}{2}=v$

$I=(x^2-2x)\left(-2\cos\frac{x}{2}\right)-\int -2\cos\frac{x}{2}(2x-2)dx$

$I=-2(x^2-2x)\cos\frac{x}{2}+2\int (x-1)\cos\frac{x}{2}dx$

$u=x-1, du=dx; 2\sin\frac{x}{2}=v$

$I=-2(x^2-2x)\cos\frac{x}{2}+4\left((x-1)2\sin\frac{x}{2}-\int 2\sin\frac{x}{2}dx\right)$

$I=-2(x^2-2x)\cos\frac{x}{2}+8(x-1)\sin\frac{x}{2}-8(-2)\cos\frac{x}{2}+2$

$I=-2(x^2-2x-8)\cos\frac{x}{2}+8(x-1)\sin\frac{x}{2}+C$

19)[испит] $\int \arccos^2 x\,dx$

$u=\arccos^2 x\,, dx=dv;\ du=2\arccos x\frac{-1}{\sqrt{1-x^2}}dx\quad x=v$

$I=x\cdot\arccos^2 x+2\int\arccos x\frac{1}{\sqrt{1-x^2}}dx$

$u=\arccos x\,, dv=\frac{-1}{\sqrt{1-x^2}}dx;\ \frac{1}{\sqrt{1-x^2}}dx=-dv;\ -dz=dv, -z=v, -\sqrt{1-x^2}=v$

$\sqrt{1-x^2}=z,\qquad \frac{1}{2\sqrt{1-x^2}}(-2x)dx=dz,\qquad \frac{xdx}{\sqrt{1-x^2}}=-dz$

$I=x\cdot\arccos^2 x+2\cdot\left(-\sqrt{1-x^2}\cdot\arccos x-\int 1\cdot dx\right)$

$I=x\cdot\arccos^2 x-2\sqrt{1-x^2}\arccos x-2x+C$

20)[испит] $\int\frac{xdx}{\cos^2 x}=x\cdot tg\ x-\int tg\ x\ dx=x\cdot tg\ x-\int\frac{\sin x}{\cos x}dx=$

$u=x, du=dx; \frac{dx}{\cos^2 x}=dv, tg\ x=v\quad ,\qquad \cos x=t, -\sin x\,dx=dt$

$I=x\cdot tg\ x+\int\frac{dt}{t}=x\cdot tg\ x+\ln|t|+C=x\cdot tg\ x+\ln|\cos x|+C$

21) $I=\int\sqrt{a^2-x^2}dx=\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx=\int\frac{a^2}{\sqrt{a^2-x^2}}dx-\int\frac{x^2}{\sqrt{a^2-x^2}}dx=$

$$I = u^2 \arcsin\frac{x}{a} - \int x \frac{x}{\sqrt{a^2-x^2}} dx$$

$$x = u, dx = du;\ \ \frac{x}{\sqrt{a^2-x^2}} dx = dx, -dz = dv, -\sqrt{a^2-x^2} = v, \frac{1}{2\sqrt{a^2-x^2}}(-2x)dx = dz$$

$$I = a^2 \arcsin\frac{x}{a} - \left(-x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2} dx\right) + C_1$$

$$I = a^2 \arcsin\frac{x}{a} + x\sqrt{a^2-x^2} - I + C_1$$

$$2I = a^2 \arcsin\frac{x}{a} + x\sqrt{a^2-x^2} + C_1$$

$$I = \frac{1}{2}a^2 \arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + C$$

22)[испит] $I = \int e^{2x} \cos\frac{x}{3} dx$

$$u = \cos\frac{x}{3}, du = -\frac{1}{3}\sin\frac{x}{3};\ \ e^{2x}dx = dv, \frac{1}{2}e^{2x} = v$$

$$I = \frac{1}{2}e^{2x} \cdot \cos\frac{x}{3} + \frac{1}{6}\int e^{2x} \cdot \sin\frac{x}{3} dx$$

$$u = \sin\frac{x}{3}, du = \frac{1}{3}\cos\frac{x}{3}dx;\ \ e^{2x}dx = dv, \frac{1}{2}e^{2x} = v$$

$$I = \frac{1}{2}e^{2x} \cdot \cos\frac{x}{3} + \frac{1}{6}\left(\frac{1}{2}e^{2x}\sin\frac{x}{3} - \frac{1}{6}\int e^{2x} \cdot \cos\frac{x}{3} dx\right)$$

$$I = \frac{1}{2}e^{2x} \cdot \cos\frac{x}{3} + \frac{1}{12}e^{2x}\sin\frac{x}{3} - \frac{1}{36} \cdot I$$

$$\frac{37}{36}I = \frac{1}{12}e^{2x}\left(6\cos\frac{x}{3} + \sin\frac{x}{3}\right) + C_1$$

$$\int R(x)dx$$

$E1.$ $\int \frac{1}{x-4} dx = \ln|x-4| + C$

$E2.$ $\int \frac{1}{(x-4)^6} dx = -\frac{1}{5(x-4)^5} + C$

$E3.$ $\int \frac{2x-5}{2x^2-5x+8} dx \approx 16$

$E4.$ $\int \frac{2x-5}{(2x^2-5x+8)^5} dx =$ решава се парцијалном интеграцијом $=$

$$= \int \frac{P_m(x)}{Q_m(x)} dx = \int R(x)dx$$

$K_1 : u \geq m \Longrightarrow P_m(x)$ поделити са $Q_m(x)$

$K_2 : Q_m(x)$ раставити на просте чиниоце

$K_3 :$ Прави рационални део представити као збир елементарних рационалних ф − ја

(1) $\frac{A}{(x-a)^k}$ (2) $\frac{Ax+B}{(ax^2+bx+c)^k}; k \in N;\ \ b^2 - 4ac < 0$

(2) $(x-a)^k \to \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$

$$(ax^2+bx+c)^k \to \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

23) [колок. 2] $\int \frac{5x-14}{x^2-6x+8}dx$

$x_{1,2} = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm 2}{2} = 3 \pm 1 \quad x_1 = 2, x_2 = 2 \quad !$ [није као проблем $E3$.]

$K_2:\ x^2-6x+8 = (x-2)(x-4)$

$K_3:\ \frac{5x-14}{x^2-6x+8} = \frac{A}{x-2} + \frac{B}{x-4} \quad /(x-2)(x-4)$

$5x-14 = A(x-4) + B(x-2)$

$(I)\ x = 4 \quad 5 \cdot 4 - 14 = B \cdot 2 \iff 2B = +6 \implies B = 3$

$x = 2 \quad 5 \cdot 2 - 14 = A(2-4) \iff -2A = -4 \implies A = 2$

$(II)\ 5x-14 = Ax - 4A + Bx - 2B$

$5x-14 = (A+B)x + (-4A-2B)$

$x^1 \quad A+B$

$x^0 \quad -4A-2B$

$-4 = -2A \implies A = 2$

$5 = 2 + B \implies B = 3$

$I = \int \left(\frac{2}{x-2} + \frac{3}{x-4}\right)dx = 2\ln|x-2| + 3\ln|x-4| + C$

24) [колок.] $\int \frac{x^2+2}{(x-1)(x+1)^2}dx$

$\frac{x^2+2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad /(x-1)(x+1)^2$

$(1)\ x^2+2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

$x = -1:\ 3 = -2C \quad C = -\frac{3}{2}$

$x = 1:\ 3 = A \cdot 4 \quad A = \frac{3}{4}$

$[(1)]'_x:\ 2x = 1 - 2(x+1) + B(x+1) + B(x-1) + C$

$x = -1:\ -2 = -2B + C \quad -2 = -2B - \frac{3}{2} \quad B = \frac{1}{4}$

$I = \int \left(\frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+1} - \frac{-\frac{3}{2}}{(x+1)^2}\right)dx = \frac{3}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| + \frac{3}{2} \cdot \frac{1}{x+1} + C$

25) [Испит] $I = \int \frac{2x^2+3x-4}{x^3-2x^2+x-2}dx$

$K_1: -$

$K_2:\ x^3-2x^2+x-2 = x^2(x-1+x-2) = (x-2)(x^2+1)$

$\frac{2x^2+3x-4}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+1)} \quad /(x-2)(x^2+1)$

$2x^2+3x-4 = A(x^2+1) + (Bx+C)(x-2)$

$x = 2:\ 10 = A(4+1) \iff 5A = 10 \implies A = 2$

$\sqrt{-1} = i \quad i^2 = -1$

$x^2+1 = 0$

$x^2 = -1$

$x = \pm\sqrt{-1} \quad x = \pm i$

$x = i : 2(-1) + 3i - 4 = (Bi + C)(i - 2)$

$3i - 6 = -B = 2C + (C - 2B)i$

$C - 2B = 3 \quad /2$

$\underline{-2C - B = -6}$

$-5B = 0 \quad C = 3 \quad B = 0$

$I = \int \left(\frac{2}{x-2} + \frac{3}{x^2+1}\right) dx = 2\ln|x-2| + 3arctg\ x + C$

26) $\int \frac{x^5 + x^4 - 8}{x^3 - 4x} dx$

$K_1: (x^5 + x^4 - 8):(x^3 - 4x) = x^2 + x + 4 + \frac{4x^2 + 16x - 8}{x^3 - 4x}$

$K_2: x^3 - 4x = x \cdot (x^2 - 4) = x \cdot (x-2)(x+2)$

$K_3: \frac{4x^2 + 16x - 8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \quad /x(x-2)(x+2)$

$4x^2 + 16x - 8 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$

$x = 0 : \ -8 = -4A \quad A = 2$

$x = 2 : \ 40 = 8B \quad B = 5$

$x = -2 : \ -24 = 8C \quad C = -3$

$I = \int \left(x^2 + x + 4 + \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2}\right) dx =$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + 2\ln|2| + 5\ln|x-2| - 3\ln|x+2| + C$

Интеграција ирационалних функција

27) $\int \frac{x + \sqrt[3]{x^2} + \sqrt[5]{x}}{x(1 + \sqrt[3]{x})} dx \qquad x = t^6, \qquad dx = 6t^5 dt \quad t = \sqrt[6]{x}$

$I = \int \frac{t^6 + t^4 + t}{t^6(1+t^2)} 6t^5 dt = 6\int \frac{t(t^5 + t^3 + 1)}{t(1+t^2)} dt = 6\int \left(t^3 + \frac{1}{t^2+1}\right) dt$

$(t^5 + t^3 + 1):(t^2 + 1) = t^3 + \frac{1}{t^2+1}$

$= 6 \cdot \frac{1}{4}t^4 + 6arctg\ t + C = \frac{3}{2}\sqrt[6]{x^4} + 6arctg\sqrt[6]{x} + C = \frac{3}{2}\sqrt[3]{x^2} + 6arctg\sqrt[6]{x} + C$

28) [Испит] $I = \int \frac{x\sqrt[3]{x+2}}{x + \sqrt[3]{x+2}} dx$

$x + 2 = t^3 \quad x = t^3 - 2 \quad dx = 3t^2 dt$

$I = \int \frac{(t^3 - 2)t}{t^3 - 2 + t} 3t^2 dt = 3\int \frac{t^6 - 2t^3}{t^3 + t - 2} dt$

$K_1: (t^6 - 2t^3):(t^3 + t - 2) = t^3 - t + \frac{t^2 - 2t}{t^3 + t - 2}$

$K_2: t^3 + t = 2$

$-1, 1, -2, 2; \quad t_1 = 1$

$(t^3 + t - 2):(t - 1) = t^2 + t + 2$

$(t^3 + t - 2) = (t - 1)(t^2 + t + 2)$

$t^2 + t + 2 = 0$

$t_{1,2} = \frac{1 \pm \sqrt{1-8}}{2} \quad \notin R$

$$K_3: \frac{t^2-2t}{(t-1)(t^2+t+2)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+2} \quad /(t-1)(t^2+t+2)$$

$$t^2-2t = A(t^2+t+2) + Bt^2 + Ct - Bt - C$$

$$t^2-2t = (A+B)t^2 + (A-B+C)t + (2A-C)$$

$$t: 1 = A+B$$

$$t: -2 = A-B+C$$

$$t: 0 = 2A-C \implies C = 2A$$

$$-2 = A-1+A+2A$$

$$4A = -1 \quad A = -\frac{1}{4} \quad B = \frac{3}{4} \quad C = -\frac{1}{2}$$

$$I = 3\int\left(t^3 - t + \frac{-\frac{1}{4}}{t-1} + \frac{\frac{3}{4}t - \frac{1}{2}}{t^2+t+2}\right)dt$$

$$I = 3\left(\frac{1}{4}t^4 - \frac{1}{4}t^2 - \frac{1}{4}\ln|t-1| + \frac{1}{4}\underbrace{\int\frac{5t-2}{t^2+t+2}dt}_{I_3}\right) + C$$

$$I_1 = \int\frac{2t+1}{t^2+t+2}dt = \ln|t^2+t+2| + C_1$$

$$I_2 = \int\frac{1}{t^2+t+2}dt = \int\frac{1}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}dt = \frac{2}{\sqrt{7}}arctg\frac{2t+1}{\sqrt{7}} + C_2$$

$$I_3 = \int\frac{3t-2}{t^2+t+2}dt = 5\int\frac{t-\frac{2}{5}}{t^2+t+2}dt = \frac{5}{2}\int\frac{2t-\frac{4}{5}}{t^2+t+2}dt = \frac{5}{2}\int\frac{2t+1-1-\frac{4}{5}}{t^2+t+2}dt$$

$$I_3 = \frac{5}{2}I_1 - \frac{3}{2}I_2$$

$$I_3 = \frac{5}{2}\ln|t^2+t+2| - \frac{3}{\sqrt{7}}arctg\frac{2t+1}{\sqrt{7}} + C_3$$

$$I = \frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4}\ln|t-1| + \frac{15}{8}\ln|t^2+t+2| - \frac{9}{4\sqrt{7}}arctg\frac{2t+1}{\sqrt{7}} + C$$

$$I = \frac{3}{4}\sqrt[3]{(x+2)^4} - \frac{3}{2}\sqrt[3]{(x+2)^2} - \frac{3}{4}\ln|x-2-1| + \frac{15}{8}\ln\left|\sqrt[3]{(x+2)^2} + x - 2 + 2\right| - \frac{9}{4\sqrt{7}}arctg\frac{2(x-2)+1}{\sqrt{7}} + C$$

...

1) $y = \frac{x^3}{4(2-x)^2}$

2) $y = x\sqrt{(x+1)^2}$

3) $y = \frac{x^2-4}{x}e^{-\frac{x}{3x}}$

4) $y = \frac{5x^2+42x+77}{x^2+7x+14}$

5) $y = \frac{x^2+x-1}{x^2+2x-1}$

6) $y = \frac{4+x+2x^2}{(x-2)^2}$

7) $y = \frac{x^3}{x-1}$

8) $y = \frac{x^4}{x^3+2}$

9) $y = 3x + \frac{6}{x} - \frac{1}{x^3}$

10) $y = \frac{x^2+2x-3}{x}e^{\frac{1}{x}}$

11) $y = (x^2-2)e^{2x}$

12) $y = x^2 - 2\ln x$

13) $y = \frac{x+2}{\sqrt{x^2+2}}$

1) $\int \frac{e^x+e^{3x}}{1-e^{2x}+e^{4x}}dx$ 5) $\int \frac{xa+arctg\,x}{\sqrt{1+x^3}}dx$ 9) $\int (2x+1)^{arctg\,x}dx$

2) $\int (x^3+x)e^{-x^2}dx$ 6) $\int \frac{\sin x-\cos x}{\sin^2 x}e^x dx$ 10) $\int \frac{\ln(1-x+x^2)}{x^2}dx$

3) $\int \frac{(x\ln x)^2}{\sqrt{x}}dx$ 7) $\int \frac{\ln x-1}{\ln^2 x}dx$

4) $\int \arcsin\sqrt{x}\,dx$ 8) $\int \frac{x^3+2x^2-2x-2}{x^2(1+x)^2}dx$

29) [Испит] $I=\int \sqrt{\frac{1-x}{x^2(1+x)}}dx$

$I=\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx, \quad \frac{1}{x}=t \Rightarrow x=\frac{1}{t} \quad -\frac{1}{x^2}dx=dt \quad \frac{1}{x^2}dx=-dt$

$I=\int \sqrt{\frac{1-\frac{1}{t}}{1+\frac{1}{t}}}(-dt)=-\int \sqrt{\frac{\frac{t-1}{t}}{\frac{t+1}{t}}}dt=-\int \sqrt{\frac{t-1}{t+1}}dt=-\int \sqrt{\frac{t-1}{t+1}\cdot\frac{t-1}{t-1}}dt$

$I=-\int \frac{t-1}{\sqrt{t^2-1}}dt=-\int \frac{t}{\sqrt{t^2-1}}dt+\int \frac{1}{\sqrt{t^2-1}}dt$

$\sqrt{t^2-1}=z, \quad \frac{1}{2\sqrt{t^2-1}}2tdt=dz, \quad \frac{t}{\sqrt{t^2-1}}dt=dz$

$I=-\int dz+\ln\left|t+\sqrt{t^2-1}\right|=-z+\ln\left|t+\sqrt{t^2-1}\right|+C$

$I=-\sqrt{t^2-1}+\ln\left|t+\sqrt{t^2-1}\right|+C=-\sqrt{\frac{1}{x^2}-1}+\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right|+C$

30) [Испит] $I=\int \frac{dx}{x\sqrt{x^2+1}}=\int \frac{xdx}{x^2\sqrt{x^2+1}}=\int \frac{tdt}{(t^2-1)t}=\int \frac{1}{t^2-1}dt$

$\sqrt{x^2+1}=t, \quad x^2+1=t^2 \Rightarrow x^2=t^2-1, \quad 2xdx=2tdt, \quad xdx=tdt$

$\frac{1}{(t-1)(t+1)}=\frac{A}{t-1}+\frac{B}{t+1} \quad /(t-1)(t+1)$

$1=A(t+1)+B(t-1)$

$t=1 \quad 1=A2, \quad t=-1 \quad 1=B(-1)$

$I=\frac{1}{2}(\ln|t-1|-\ln|t+1|)+C=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|+C$

$I=\frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\cdot\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}-1}\right|+C=\frac{1}{2}\ln\left|\frac{\left(\sqrt{x^2+1}-1\right)^2}{x^2}\right|+C=\ln\left|\frac{\sqrt{1+x^2}-1}{x}\right|+C$

може и преко $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

Интеграли рационалних тригонометријских ф-ја

$I=\int R(\sin x,\cos x)dx$

I (1) $tg\frac{x}{2}=t, \quad I=\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\cdot\frac{2}{1+t^2}dt$

$$II\begin{cases}(2)\quad R(-\sin x,\cos x)=-R(\sin x,\cos x);\ \cos x=t\\(3)\quad R(\sin x,-\cos x)=-R(\sin x,\cos x);\ \sin x=t\\(4)\quad R(-\sin x,\cos x)=R(\sin x,\cos x);\ \operatorname{tg}x=t\end{cases}$$

1) $\displaystyle\int\frac{1}{3\sin x+4\cos x+5}dx \qquad tg\frac{x}{2}=t$

$$I=\int\frac{1}{3\cdot\frac{2t}{1+t^2}+4\cdot\frac{1-t^2}{1+t^2}+5}\cdot\frac{1}{1+t^2}dt=2\cdot\int\frac{1}{6t+4-4t^2+5+5t^2}dt=$$

$$=2\cdot\int\frac{1}{t^2+6t+9}dt=2\cdot\int\frac{1}{(t+3)^2}dt=2\cdot\frac{-1}{t+3}+C=-\frac{2}{tg\frac{x}{2}+3}+C$$

2) $\displaystyle\int\frac{1}{\cos^2 x}dx \qquad R(\sin x,-\cos x)=-R(\sin x,\cos x);\quad \sin x=t,\qquad \cos x\,dx=dt$

$$\int\frac{1}{\cos^2 x}dx=\int\frac{\cos x}{\cos^4 x}dx=\int\frac{\cos x}{(\cos^2 x)^2}dx=\int\frac{dt}{(1-t^2)^2}=\int\frac{dt}{(t^2-1)^2}$$

$$\frac{1}{(t-1)^2(t+1)^2}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2}\quad /(t-1)^2(t+1)^2$$

$$1=A(t-1)(t+1)^2+B(t+1)^2+C(t+1)(t-1)^2+D(t-1)^2$$

$t=1\quad 1=4B\Rightarrow B=\frac{1}{4}$

$t=-1\quad 1=4D\Rightarrow D=\frac{1}{4}$

$$[(1)]'_t\quad 0=A(t+1)^2+A(t-1)2(t+1)+2B(t+1)+C(t-1)^2+C(t+1)^2 2(t-1)+2D(t-1)$$

$t=1\quad 0=4A+4B$

$\qquad 0=4A+1\Rightarrow A=-\frac{1}{4}$

$t=-1\quad 0=4C-4D$

$\qquad 0=4C+1\Rightarrow C=-\frac{1}{4}$

$$I=\int\left(\frac{-\frac{1}{4}}{t-1}+\frac{-\frac{1}{4}}{t+1}+\frac{\frac{1}{4}}{(t+1)^2}\right)dt=-\frac{1}{4}\ln|t-1|-\frac{1}{4}\cdot\frac{1}{t-1}-\frac{1}{4}\ln|t+1|-\frac{1}{4}\cdot\frac{1}{t+1}+C$$

$$I=-\frac{1}{4}\ln|(t-1)(t+1)|-\frac{1}{4}\cdot\frac{t+1+t-1}{(t-1)(t+1)}+C=-\frac{1}{4}\ln|t^2-1|-\frac{1}{4}\cdot\frac{2t}{t^2-1}+C$$

$\sin x=t,\qquad \sin^2 x=t^2,\qquad \sin^2 x-1=t^2-1,\qquad -\cos^2 x=t^2-1$

$$I=-\frac{1}{4}\ln|\cos^2 x|-\frac{1}{2}\frac{\sin x}{-\cos^2 x}+C=-\frac{1}{2}\ln|\cos x|+\frac{1}{2}\frac{\sin x}{\cos x}+C$$

3) $\displaystyle J=\int\frac{1}{\sin x\cos^3 x}dx=\int\frac{1}{\sin x\cos x}\cdot\frac{dx}{\cos^2 x}=$

$\qquad R(-\sin x,\cos x)=R(\sin x,\cos x);\ \operatorname{tg}x=t,\qquad \frac{1}{\cos^2 x}dx=dt$

$$=\int\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\cdot\frac{dx}{\cos^2 x}=\int(tg\,x+ctg\,x)\frac{dx}{\cos^2 x}=\int\left(t+\frac{1}{t}\right)dt=\frac{1}{2}t^2|t|+C=$$

$$=\frac{1}{2}tg^2\,x+\ln|tg\,x|+C$$

4) $\displaystyle\int\frac{1}{\sin x\,(1+\cos x)}dx$

$R(-\sin x, \cos x) = -R(\sin x, \cos x);\ \cos x = t, \qquad -\sin x\, dx = dt, \qquad \sin x\, dx = -dt$

$$I = \int \frac{\sin x}{\sin^2 x\,(1+\cos x)} dx = \int \frac{-dt}{(1-t^2)(1+t)} = \int \frac{dt}{(t^2-1)(t+1)}$$

$$\frac{1}{(t-1)(t+1)^2} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{(t+1)^2}$$

$1 = A(t+1)^2 + B(t-1)(t+1) + C(t-1)$

$1 = (A+B)t^2 + (2A+C)t + A - B - C$

$t^2 : A + B = 0 \Longrightarrow B = -A$

$t^1 : 2A + C = 0 \Longrightarrow C = -2A$

$t^0 : A - B + C = 1 \Longrightarrow A + A + 2A = 1$

$$A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

$$I = \int \left(\frac{\frac{1}{4}}{t-1} + \frac{-\frac{1}{4}}{t+1} + \frac{-\frac{1}{2}}{(t+1)^2} \right) dt = \frac{1}{4}\ln|t-1| - \frac{1}{4}\ln|t+1| + \frac{1}{2}\cdot\frac{1}{t+1} + C$$

$$I = \frac{1}{4}\ln|\cos x - 1| - \frac{1}{4}\ln|\cos x + 1| + \frac{1}{2}\cdot\frac{1}{1+\cos x} + C$$